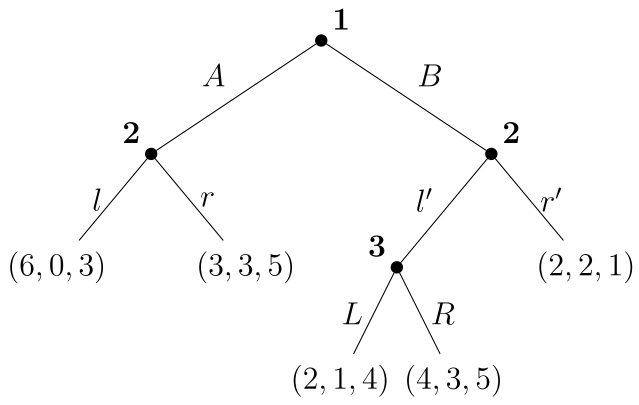
PLEASE ANSWER ALL QUESTIONS.
PLEASE EXPLAIN YOUR ANSWERS.

1. Consider the extensive-form game given by the following game tree (the first payoff is that of player 1, the second payoff that of player 2, etc.):




(a) Answer the following questions.

i. Is this a game of perfect or imperfect information? How many proper subgames are there (excluding the game itself)? What are the strategy sets of the three players?

ii. Find all (pure strategy) Subgame-perfect Nash Equilibria. Argue why you use the solution method you use.

iii. Is the strategy profile $(A, rr', L)$ a Nash Equilibrium? Discuss briefly (max. 3 sentences).

**Solution:** Perfect information. 3 proper subgames. $S_1 = \{A, B\}$. $S_2 = \{ll', lr', rl', rr'\}$. $S_3 = \{L, R\}$. Since perfect, complete information, we can solve by backward induction to get $SPNE = \{(B, rl', R)\}$. The strategy profile $(A, rr', L)$ is NE but rests on off-equilibrium-path 'threats' that are not credible.

(b) Consider again the game in (a), but suppose now that player 2 does not observe the move of player 1.

i. Draw the resulting game tree.

ii. Is this a game of perfect or imperfect information? How many proper subgames are there (excluding the game itself)? What are the strategy sets of the three players?

iii. Find all (pure strategy) Subgame-perfect Nash Equilibria. Compare to the solution in (a).

**Solution:** Game tree obtained by connecting P2's two nodes in the original tree. P2 now has a single information set. There is 1 proper subgame. Strategy sets as before, except $S_2 = \{l, r\}$. P3's subgame gives $s_3 = R$. Any SPNE must have $s_3 = R$, and thus we can substitute this into the game, yielding:

| | | Player 2 | |
|---|---|---|---|
| | | $l$ | $r$ |
| Player 1 | $A$ | $\underline{6}, 0$ | $\underline{3}, \underline{3}$ |
| | $B$ | $4, \underline{3}$ | $2, 2$ |

Thus: $SPNE = \{(A, r, R)\}$. Now, P1 has an incentive to deviate to $A$ if P2 plays $l$, making it impossible to have a SPNE where P1 plays $B$.

2. Consider a *second-price sealed bid auction* with two bidders, who have valuations $v_1$ and $v_2$, respectively.

   Assume that the values are distributed independently uniformly with

   $$v_i \sim u(0, 1).$$

   Thus, the values are **private**. Show that there is a symmetric Bayesian Nash Equilibrium where the players bid their valuation: $b_i(v_i) = v_i$ (recall that the auction format is second-price sealed bid).

**Solution:** Throughout suppose that $j$ keeps to his equilibrium strategy: $b_j = v_j$. The probability that two bids are the same is zero, and therefore we only consider 'inequalities'.

Suppose player $i$ deviates by bidding $b' < v_i$. If $v_j > v_i$ then $b' < b_j$ and player $i$ loses in either case. If $v_j < b' < v_i$ then player $i$ wins and pays $p = v_j$ in either case. If $b' < v_j < v_i$ then player $i$ wins and gets payoff $v_i - v_j > 0$ if he sticks to the equilibrium strategy, and he loses and gets payoff 0 if he deviates. Thus, $b' < v_i$ is never a profitable deviation.

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Hence, bidding $b_j$ is weakly optimal for both players, and therefore a NE.

3. Three entrepreneurs are considering starting a new tech company. They are free to form a company of any size between themselves. Entrepreneurs A and B are very experienced, with A being slightly more experienced than B, whereas entrepreneur C has no experience whatsoever. If entrepreneurs A and B work together in the company, the value of the company is 5 dollars (regardless of whether entrepreneur

C joins the company). If entrepreneur A starts the company alone or with C, it is worth 2 dollars. If entrepreneur B starts the company alone it is worth 0 dollar, but if B starts it with C, it is worth 1 dollar. If entrepreneur C starts the company alone, it is worth 0 dollar.

(a) Think of this situation as a coalitional game with transferable payoffs. Write down the value of each coalition.

> **Solution:** The values are:
>
> $$\begin{aligned} V(ABC) = V(AB) &= 5, \\ V(AC) = V(A) &= 2, \\ V(BC) &= 1, \\ V(B) = V(C) &= 0. \end{aligned}$$

(b) Find the core of this game.

> **Solution:** Feasibility implies $V_A + V_B + V_C \leq 5$, and the coalition values give us the following restrictions:
>
> $$\begin{aligned} V_A + V_B + V_C &\geq 5, \\ V_A + V_B &\geq 5, \\ V_A + V_C &\geq 2, \\ V_B + V_C &\geq 1, \\ V_A &\geq 2, \\ V_B &\geq 0, \\ V_C &\geq 0. \end{aligned}$$
>
> Since $V_A + V_B \geq 5$, then the core (if it is non-empty) must have $V_A + V_B = 5$ and $V_C = 0$. Furthemore, if $V_A \geq 2$ and $V_B \geq 1$, this satisfies all the restrictions, whereas if these inequalities do not hold, either $A$ or $B$ can 'deviate'.
> Hence, the core is equal to $\{(V_A, V_B, V_C) = (2 + v, 3 - v, 0), v \in [0, 2]\}$.

(c) If all the entrepreneurs obtain a strictly positive payoff in the core explain why this is. If some entrepreneur receives a zero payoff in the core, explain why this is.

> **Solution:** Entrepreneur C receives a zero payoff because A and B can generate the maximum value of the firm alone, and C cannot generate any value on his own. Thus, even though C could generate value in some configurations of the firm, A and B do not need him and therefore he ends up with zero payoff.

4. Consider the following version of Spence's education signaling model, where a firm is hiring a worker. Workers are characterized by their type $\theta$, which measures their ability. There are two worker types: $\theta \in \{\theta_L, \theta_H\}$. Nature chooses the worker's type, with $\mathbb{P}(\theta = \theta_H) = p$ and $\mathbb{P}(\theta = \theta_L) = 1 - p$. Assume $p \in (0, 1)$. The worker observes his own type, but the firm does not.

   The worker can choose his level of education: $e \in \mathbb{R}^+$. The cost to him of acquiring this education is

$$c_\theta(e) = \frac{e}{\theta}.$$

   Education is observed by the firm, who then forms beliefs about the worker's type: $\mu(\theta|e)$. We assume that the marginal productivity of a worker is equal to his ability $\theta$ and that the firm is in competition such that it pays the marginal productivity: $w(e) = \mathbb{E}(\theta|e)$. Thus, the payoff to a worker conditional on his type and education is

$$u_\theta(e) = w(e) - c_\theta(e).$$

   Suppose for this exercise that $\theta_H = 4$ and $\theta_L = 2$.

   (a) In a separating equilibrium the low-ability worker chooses education level $e_L$ and obtains wage $w_L = w(e_L)$. Is it possible that $e_L > 0$? Explain briefly (max. 3 sentences).

   > **Solution:** No. Suppose $e_L > 0$. In a separating equilibrium, $L$ gets $2 - e_L/2$. For any beliefs we have $u_L(0) = w(0) \geq \theta_L = 2 > 2 - e_L/2$: a profitable deviation exists.

   (b) Find a separating pure strategy Perfect Bayesian Equilibrium where the two types choose education levels $e_L$ and $e_H$, respectively, and the low ability type is indifferent between choosing $e_L$ and $e_H$. Assume that off the equilibrium path, the firm assigns zero probability to the worker being type $\theta_H$.

   > **Solution:** By assumption, $\mu(\theta_H|e)$ is equal to 1 if $e = e_H$ and equal to 0 otherwise. Thus, $w(e)$ is equal to 4 when $e = e_H$ and equal to 2 otherwise. We argued above that $e_L = 0$ in equilibrium. Given $w(e)$, $e = e_L = 0$ strictly dominates all $e \neq e_H$ for both types. Hence, only the strategies $e_L$ and $e_H$ need to be considered. To make the low type indifferent:
   >
   > $$u_L(0) = u_L(e_H) \Leftrightarrow 2 = 4 - \frac{e_H}{2} \Leftrightarrow e_H = 4.$$
   >
   > Clearly, the high type prefers $e_H = 4$ as for any $e' \neq e_H$, we have
   >
   > $$u_H(e') = 2 - \frac{e'}{4} \leq 2 < 4 - \frac{4}{4} = u_H(e_H).$$
   >
   > Hence: the specified $w(e)$ and $\mu(\cdot|e)$, together with $e_L = 0$ and $e_H = 4$ form a PBE.

   (c) Find a pooling pure strategy Perfect Bayesian Equilibrium in which both types choose education level $\bar{e}$, and the low ability type is indifferent between choosing

$e = 0$ and $e = \bar{e}$. Assume that off the equilibrium path, the firm assigns zero probability to the worker being type $\theta_H$. Does the pooling equilibrium of (c) satisfy SR6? You can show this either graphically or algebraically.

**Solution:** $\mu(\theta_H|e)$ is equal to $p$ if $e = \bar{e}$ and equal to 0 otherwise. Thus, $w(e)$ is equal to $p(4) + (1-p)(2) = 2(1+p)$ when $e = \bar{e}$ and equal to 2 otherwise. Given $w(e)$, $e = 0$ strictly dominates all $e \neq \bar{e}$ for both types. Hence, only these two strategies need to be considered. To make the low type indifferent:

$$u_L(0) = u_L(\bar{e}) \Leftrightarrow 2 = 2(1+p) - \frac{\bar{e}}{2} \Leftrightarrow \bar{e} = 4p.$$

Clearly, the high type prefers $e = \bar{e} = 4p$ over $e = 0$ as

$$u_H(\bar{e}) = 2(1+p) - \frac{\bar{e}}{4} = 2(1+p) - \frac{4p}{4} = 2 + p \geq 2 = u_H(0)$$

holds. Hence: the specified $w(e)$, $\mu(\cdot|e)$, together with $\bar{e} = p$ form a PBE.

*Checking SR6*: For the low-ability type, the equilibrium strategy strictly dominates $e$ whenever

$$2(1+p) - \frac{\bar{e}}{2} > 4 - \frac{e}{2} \Leftrightarrow 2 > 4 - \frac{e}{2} \Leftrightarrow e > 4.$$

For the high-ability type, the equilibrium strategy strictly dominates $e$ whenever

$$2(1+p) - \frac{\bar{e}}{4} > 4 - \frac{e}{4} \Leftrightarrow 2 + p > 4 - \frac{e}{4} \Leftrightarrow e > 4(2-p) > 4.$$

Hence, $e \in (4, 4(2-p))$ are equilibrium dominated for $L$ but not for $H$. SR6: $\mu(\theta_H|e) = 1$ and hence $w(e) = 4$ for $e \in (4, 4(2-p))$. The pooling equilibrium does not satisfy SR6.